LECTURE XIX

Integral Convergence Tests

Can we compute the following integral?

$$\int_1^\infty e^{-x^2} dx$$

We don't know its antiderivative. No good substitution will work. Integration by parts will only increase its complexity. Indeed,

$$u = e^{-x^2}, v' = 1$$
$$du = -2xe^{-x^2}, v = x$$

and thus integration by parts yields

$$\int e^{-x^2} dx = xe^{-x^2} + 2\int x^2 e^{-x^2} dx$$

where the second integral looks more difficult to solve.

Can we lower our standards and just ask whether this integral will yield a finite value or not? It looks like the integrand is decaying. Maybe it will be finite. How do we show it? We compare to another function that dominates it in the domain and show the integral for the dominating function is finite. Big brothers cover small brothers.

$$\int_1^\infty e^{-x^2} dx = \lim_{b\to\infty} \int_1^b e^{-x^2} dx \le \lim_{b\to\infty} \int_1^b e^{-x} dx = \lim_{b\to\infty} -e^{-b} + e^{-1} < e^{-1} \approx 0.36788$$

independent of b. **The first inequality is because of the following:** for $x \ge 1$, we have

$$x^2 \ge x \implies -x^2 \le -x \implies e^{-x^2} \le e^{-x}$$

since e^x is a monotone increasing function, i.e. for $y \ge z$, $e^y \ge e^z$. Therefore, we know

$$\int_1^\infty e^{-x^2} dx = \lim_{b\to\infty} \int_1^b e^{-x^2} dx < 0.36788$$

which shows that the improper integral is **convergent.**

In general, for every positive function, we have the following result.

Theorem. ***(Direct Comparison Test)*** *Let f and g be continuous on $[a, \infty)$ with $0 \le f(x) \le g(x)$ for all $x \ge a$. Then*

(1) *If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.*
(2) *If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges.*

In other words, a sane big brother controls the fate of the little brother. An insane little brother messes up the big brother. (Here computer scientists should think of logic – if A then B is equivalent to if not B then not A.)

Example. Do these exercises yourself.

(1)

$$\int_1^\infty \frac{\sin^2(x)}{x^2} dx$$

(2)

$$\int_1^\infty \frac{1}{\sqrt{x^2 - 0.1}} dx$$

(3)

$$\int_0^{\pi/2} \frac{\cos(x)}{\sqrt{x}} dx$$

Another test will assist the limit comparison test, in the sense that it will only tell you whether a pair of functions either both converges or both diverges, if their ratio is fixed at infinity.

Theorem. ***(Limit Comparison Test)*** *If the positive functions f and g are continuous on $[a, \infty)$, and if*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \quad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\,dx \quad \textit{and} \quad \int_a^\infty g(x)\,dx$$

either both converge or both diverge.

The essence of this theorem is, when given $f(x)$ and you are asked to check whether $\int_a^\infty f(x)\,dx$ is finite or not, suppose you want to show that this integral diverges, then you are going to pick a $g(x)$ where $\int_a^\infty g(x)\,dx$ diverges, and at the same time the limit

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

exists. Then the choice of g and the given f satisfies the theorem conditions. Since the integral of g is already divergent as you picked it, that of f will also diverge, by the theorem conclusion.

It boils down to whether your initial hunch is eventually correct or not. But you must have an initial hunch, whether a given integral should converge or diverge.

Example. Check the convergence of

$$\int_1^\infty \frac{1 - e^{-x}}{x} dx$$

Develop the hunch: The numerator as $x \to \infty$ seems to converge to 1, so the term e^{-x} is less and less important. We know that $\int_1^\infty \frac{1}{x} dx$ is divergent. Hence we may anticipate that this integral eventually is divergent. Hence, the function $g(x)$ we pick to compare $f(x) = \frac{1-e^{-x}}{x}$ to is certainly $g(x) = \frac{1}{x}$, because if the limit evaluation of the ratio f/g goes to finite number, yet integral of g diverges, then by LCT integral of f also diverges. Indeed,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \left(\frac{1 - e^{-x}}{x}\right)\left(\frac{x}{1}\right) = \lim_{x\to\infty} 1 - e^{-x} = 1 < \infty.$$

Thus $\int_1^\infty f(x)\,dx$ diverges.

Can you do this by DCT? No, because even if you choose the dominating function $g(x) = \frac{1}{x}$, which indeed is dominating, the divergence of $\int \frac{1}{x} dx$ tells you nothing. You actually need to choose some g that is dominated by f and yet the integral of g still diverges. Then the integral of f really diverges. In this case, it is pretty hard to come up with such g. LCT is the way to go.